# Handling Fibred Algebraic Effects

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**Dependent Types**

and

**Logical Reasoning**

**Algebraic Effects**

and

**Effect Handlers**

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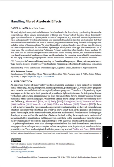
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1 INTRODUCTION

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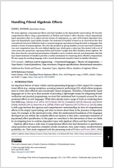


**Algebraic Effects**

and

**Effect Handlers**

**What can we do?**

**How to do it?**

# Outline

- Setting the scene
  - **Algebraic effects** and their **handlers**
  - An effectful dependently typed **core calculus** (FoSSaCS'16)

    [A., Ghani, Plotkin'16]

- What can we gain from handlers + dependent types?
  - Modular programming with handlers + expressiveness of d. types
  - **Extrinsic reasoning** about effectful computations

- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common **term-level def.** of handlers (has issues)
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# Algebraic effects

- Moggi taught us to model comp. effects using **monads** $(T, \eta, (-)^\dagger)$

$$\eta_A : A \to TA \qquad (f : A \to TB)^\dagger_{A,B} : TA \to TB$$

- Plotkin and Power showed that most of these monads arise from
  - **operation symbols** – representing the **sources** of effects

$$\mathsf{raise} : \mathsf{Exc} \longrightarrow 0 \qquad \mathsf{get} : \mathsf{Loc} \longrightarrow \mathsf{Val} \qquad \mathsf{put} : \mathsf{Loc} \times \mathsf{Val} \longrightarrow 1$$

  - **equations** – describing the computational **behaviour**

$$\ell{:}\mathsf{Loc} \mid w{:}1 \vdash \mathtt{get}_\ell(x.\mathtt{put}_{\langle \ell, x \rangle}(w(\star))) = w(\star)$$

- The algebraic approach significantly simplifies
  - **choosing** a monad/adjunction to model a given language
  - modelling **combinations** of two or more comp. effects
  - **generic** effectful programming (via **handlers**)

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# Handlers of algebraic effects

- Plotkin and Pretnar's **handlers** of algebraic effects
  - generalisation of exception handlers
  - given by **redefining** the given ops. (handlers denote **algebras**)
  - many uses – stream redirection, state, rollbacks, concurrency, ...
- Usually included in languages using the **handling** construct

$$M \texttt{ handled with } (\{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}}, \overrightarrow{W_{\mathrm{eq}}}) \texttt{ to } y \,{:}\, A \texttt{ in}_{\underline{C}}\ N_{\mathrm{ret}}$$

interpreted using the **homomorphism** $FA \longrightarrow \langle U\underline{C}, \overrightarrow{f_{N_{\mathrm{op}}}} \rangle$, i.e.,

$$(\mathrm{op}_V(y.M)) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y \,{:}\, A \texttt{ in}_{\underline{C}}\ N_{\mathrm{ret}}$$
$$=$$
$$N_{\mathrm{op}}[V/x_v][\lambda\, y \,{:}\, O \,.\, \texttt{thunk}\,(M \texttt{ handled with } \ldots)/x_k]$$

and

$$(\texttt{return}\ V) \texttt{ handled with } \{\ldots\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} \texttt{ to } y \,{:}\, A \texttt{ in}_{\underline{C}}\ N_{\mathrm{ret}} \;=\; N_{\mathrm{ret}}[V/y]$$

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# A core dependently typed effectful calculus

- Natural extension of Martin-Löf's (intensional) type theory
  - clear distinction between **values** and **computations** (CBPV, EEC)

- **Value types** ($\Gamma \vdash A$) and **computation types** ($\Gamma \vdash \underline{C}$)

$$A, B \;::=\; \ldots \;\mid\; U\underline{C} \qquad \underline{C}, \underline{D} \;::=\; FA \;\mid\; \Pi\, x\!:\!A\,.\,\underline{C} \;\mid\; \boxed{\Sigma\, x\!:\!A\,.\,\underline{C}}$$

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# The calculus we propose in this paper . . .

- . . . is a variant of the FoSSaCS'16 calculus, with
  - a Tarski-style **value universe** $\mathcal{U}$
    - with **codes** written as $\widehat{\Pi}$ , $\widehat{\Sigma}$ , $\widehat{0}$ , $\widehat{1}$ , . . .
    - but thinking of them as $\forall$ , $\exists$ , $\bot$ , $\top$ , . . .
  - fibred **algebraic effects**
    - dep. typed **operation symbols** op : $(x_v : I) \longrightarrow O$
    - ops. determine **computation terms** $\mathrm{op}^{\underline{C}}_V(y : O[V/x_v] \,.\, M)$
    - effect equations determine **definitional equations**
  - a **derivable** "into-comps." variant of **handlers** and **handling**

$$M \texttt{ handled with } (\{\mathrm{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}} ; \overrightarrow{W_{\mathrm{eq}}}) \texttt{ to } y : A \texttt{ in}_{\underline{C}} \; N_{\mathrm{ret}}$$

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# Handlers are useful for extrinsic reasoning!

- An alternative to using prop. eq. on thunks for **preds. on** $M : FA$
  - With handlers we define **predicates** $P : UFA \to \mathcal{U}$ by
    1) equipping $\mathcal{U}$ (or a resp. type) with an **algebra** structure
    2) **handling** the given computation using that algebra
  - Intuitively, $P\ (\mathtt{thunk}\, M)$ computes a **proof obligation** for $M$
  - We discuss **three examples** of such predicates
- Also, an alternative to monadic reification for **rel. reasoning**
  - E.g., relating **stateful comps.** $M, N : FA$ as **functions** $S \to A \times S$
  - Not investigated in this paper
  - See [Grimm et al.'18] for **reification-based** relational reasoning

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  - With handlers we define **predicates** $P : UFA \to \mathcal{U}$ by
    1) equipping $\mathcal{U}$ (or a resp. type) with an **algebra** structure
    2) **handling** the given computation using that algebra
  - Intuitively, $P\ (\texttt{thunk}\, M)$ computes a **proof obligation** for $M$
  - We discuss **three examples** of such predicates

- Also, an alternative to monadic reification for **rel. reasoning**
  - E.g., relating **stateful comps.** $M, N : FA$ as **functions** $S \to A \times S$
  - Not investigated in this paper
  - See [Grimm et al.'18] for **reification-based** relational reasoning

# Ex1: Lifting predicates to effectful comps.

- Given a predicate $P : A \to \mathcal{U}$ on **return values**,

  we define a predicate $\Box P : UFA \to \mathcal{U}$ on **(I/O)-comps.** as

$$\Box P \stackrel{\text{def}}{=} \lambda\, y \,{:}\, UFA \,.\, (\texttt{force}\, y)\ \texttt{handled with}\ \{\ldots\}_{\mathsf{op} \in \mathcal{S}_{\mathsf{I/O}}}\ \texttt{to}\ y' \,{:}\, A\ \texttt{in}_{\mathcal{U}}\ P\, y'$$

  using the **handler** given by

$$\begin{array}{lll} \mathsf{read}(x_k) & \mapsto & \widehat{\Pi}\, y \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, x_k\ y \qquad\qquad (\text{where } x_k \,{:}\, \mathsf{Chr} \to \mathcal{U}) \\ \mathsf{write}_{x_v}(x_k) & \mapsto & x_k\ \star \qquad\qquad (\text{where } x_v \,{:}\, \mathsf{Chr},\ x_k \,{:}\, 1 \to \mathcal{U}) \end{array}$$

- $\Box P$ is similar to the **necessity modality** from Evaluation Logic

$$\Gamma \vdash \Box P\, (\texttt{thunk}\, (\texttt{read}(x \,.\, \texttt{write}_{\text{'e'}}(\texttt{return}\, V)))) = \widehat{\Pi}\, x \,{:}\, \mathsf{El}(\widehat{\mathsf{Chr}}) \,.\, P\ V$$

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# Ex2: Dijkstra's weakest precondition sem.

- Given a postcondition on **return values** and **final states**

$$Q : A \to S \to \mathcal{U} \qquad\qquad (S \stackrel{\text{def}}{=} \Pi\, \ell{:}\mathsf{Loc}\,.\mathsf{Val}(\ell))$$

  we define a precondition for **stateful comps.** on **initial states**

$$\mathsf{wp}_Q : UFA \to S \to \mathcal{U}$$

  by

  **1)** handling the given comp. into a **state-passing function** using

$$V_{\mathsf{get}}\,,\, V_{\mathsf{put}} \quad \text{on} \quad S \to \mathcal{U} \times S \qquad\qquad \text{and} \qquad\qquad V_{\mathsf{ret}} \text{ ``$=$'' } Q$$

  **2)** feeding in the **initial state**; and **3)** projecting out the **value of $\mathcal{U}$**

- Then, $\mathsf{wp}_Q$ satisfies the **expected properties**, such as

$$\Gamma \vdash \mathsf{wp}_Q\ (\mathtt{thunk}\,(\mathtt{return}\, V)) \quad = \quad \lambda\, x_S : S\,.\, Q\ V\ x_S$$

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# Ex3: Allowed patterns of (I/O)-effects

- Assuming an inductive type of **I/O-protocols**, given by

$$\mathtt{e} : \text{Protocol} \qquad \mathtt{r} : (\text{Chr} \to \text{Protocol}) \to \text{Protocol}$$

$$\mathtt{w} : (\text{Chr} \to \mathcal{U}) \times \text{Protocol} \to \text{Protocol}$$

- We can define a **relation** between **comps.** and **protocols**

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# Outline

- Setting the scene
  - **Algebraic effects** and their **handlers**
  - An effectful dependently typed **core calculus** (FoSSaCS'16)

    [A., Ghani, Plotkin'16]
- What can we gain from handlers + dependent types?
  - Modular programming with handlers + expressiveness of d. types
  - **Extrinsic reasoning** about effectful computations
- Extending the FoSSaCS'16 calculus with alg. effects and handlers
  - Take 1: The common **term-level def.** of handlers (has issues)
  - Take 2: A new **type-level treatment** of handlers

# Extending the FoSSaCS'16 calculus

- We assume given a **fibred effect theory** $\mathcal{T} = (\mathcal{S}, \mathcal{E})$
- First, we extend the calculus with **algebraic effects** as follows:
  - we extend the **computation terms** with

$$M, N \;::=\; \dots \;\mid\; \mathrm{op}^{\underline{C}}_{V}(y{:}O[V/x_v]\,.\,M) \qquad (\mathrm{op} : (x_v{:}I) \longrightarrow O \in \mathcal{S})$$

  - we extend the **equational theory** with equations given in $\mathcal{E}$
  - we capture the **interaction** of comp. terms and ops. with the eq.

$$\frac{\Gamma \vdash V : I \quad \Gamma, x{:}O[V/x_v] \vdash M : \underline{C} \quad \Gamma \mid z{:}\underline{C} \vdash K : \underline{D}}{\Gamma \vdash K[\mathrm{op}^{\underline{C}}_{V}(x.M)/z] = \mathrm{op}^{\underline{D}}_{V}(x.K[M/z]) : \underline{D}} \; (\mathrm{op} : (x_v{:}I) \longrightarrow O \in \mathcal{S})$$

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# Take 1: Term-level definition of handlers

- Begin by extending the FoSSaCS'16 **computation terms** with

$$M, N \;::=\; \ldots \;\mid\; M \texttt{ handled with } \{\texttt{op}_{x_v}(x_k) \mapsto N_{\texttt{op}}\}_{\texttt{op} \in \mathcal{S}_{\text{eff}}} \texttt{ to } y\,{:}\,A \texttt{ in}_{\underline{C}}\ N_{\text{ret}}$$

- But as handling denotes a **homomorphism**, then perhaps also

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$$\Gamma \vdash \texttt{write}_{\mathsf{a}}(\texttt{return}\,\star) = \texttt{write}_{\mathsf{z}}(\texttt{return}\,\star) : F1$$

- At a very high-level, the problem is (see the paper for details)
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# How to proceed?

- Possible ways to solve this unsoundness problem
  - **Option 1: Change** the FoSSaCS'16 calculus
    - change the equational theory of homomorphism terms
    - hom. terms would not denote homomorphisms any more
    - investigated for exceptions in CBPV with stacks by [Levy'06]
  - **Option 2:** Keep the FoSSaCS'16 calculus **unchanged**
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# Take 2: A type-level treatment of handlers

- Instead, we extend the FoSSaCS'16 **computation types** with
  - a **user-defined algebra type**

$$\underline{C}\,,\underline{D} \;::=\; \ldots \;\mid\; \langle A; \overrightarrow{V_{\mathrm{op}}}; \overrightarrow{W_{\mathrm{eq}}} \rangle$$

  where

    - $A$ is the **carrier** value type
    - $\overrightarrow{V_{\mathrm{op}}}$ is a set of user-defined **operations**
    - $\overrightarrow{W_{\mathrm{eq}}}$ is a set of **witnesses** of equational proof obligations

- As a result, we can derive the **handing construct** as

$$M \;\texttt{handled with}\; (\{\texttt{op}_{x_v}(x_k) \mapsto N_{\mathrm{op}}\}_{\mathrm{op} \in \mathcal{S}_{\mathrm{eff}}}; \overrightarrow{W_{\mathrm{eq}}})\;\texttt{to}\; y\!:\!A \;\texttt{in}_{\underline{C}}\; N_{\mathrm{ret}}$$

$$\stackrel{\mathrm{def}}{=}$$

$$\texttt{force}_{\underline{C}}(\texttt{thunk}\,(\underbrace{M \;\texttt{to}\; y\!:\!A \;\texttt{in}\; \texttt{force}_{\langle U\underline{C}; \overrightarrow{V_{N_{\mathrm{op}}}}; \overrightarrow{W_{\mathrm{eq}}} \rangle}(\texttt{thunk}\, N_{\mathrm{ret}})}_{\text{temporarily working at type } \langle U\underline{C}; \overrightarrow{V_{N_{\mathrm{op}}}}; \overrightarrow{W_{\mathrm{eq}}} \rangle}))$$

  and similarly for the "**into-values**" variant of it

# Take 2: A type-level treatment of handlers

- Instead, we extend the FoSSaCS'16 **computation types** with
  - a **user-defined algebra type**

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# Conclusion

- In conclusion
  - handlers are natural for **extrinsic reasoning** about computations
    - lifting predicates from return values to computations
    - Dijkstra's weakest precondition semantics of state
    - specifying patterns of allowed (I/O)-effects
  - they admit a principled **type-based treatment**

- See the paper for
  - **formal details** of what I have shown you today
  - families fibrations based **denotational semantics**
  - discussion about the calculus's inherent **extensional nature**
  - **Agda code** for the example predicates $P : UFA \to \mathcal{U}$

Thank you!

Questions?