When Is a Container a Comonad?

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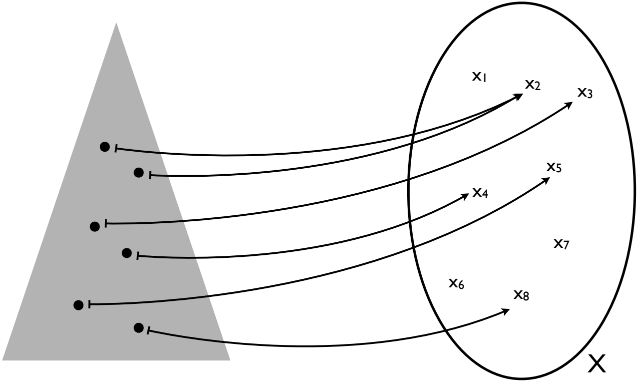
Tallinn, 28 March 2012

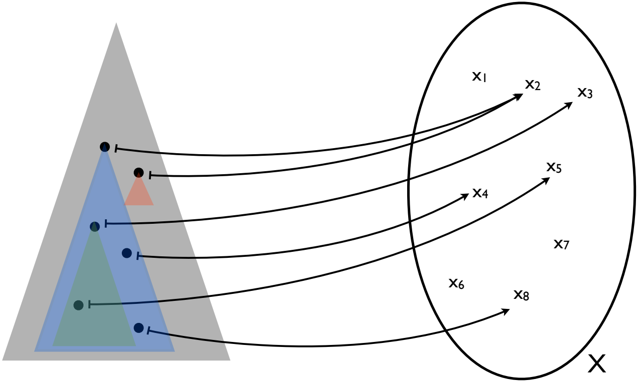
Container syntax of datatypes

- Many datatypes can be represented in terms of shapes and positions in shapes
  - Examples: lists, streams, colists, trees, etc.
  - Non-examples: sets, bags
- Containers provide us with a handy syntax to analyze such datatypes



Directing containers?

- Containers often exhibit a natural notion of subshape given by positions in shapes
- Natural questions arise:
  1. What is the appropriate specialization of containers?
  2. Does this admit a nice categorical theory?



Directed containers

- A directed container is given by
  - $S : \mathsf{Set}$ *(shapes)*
  - $P : S \to \mathsf{Set}$ *(positions)*

  and

  - $\downarrow : \Pi s : S.\, P\, s \to S$ *(subshape)*
  - $\mathsf{o} : \Pi\{s : S\}.\, P\, s$ *(root position)*
  - $\oplus : \Pi\{s : S\}.\, \Pi p : P\, s.\, P\, (s \downarrow p) \to P\, s$ *(subshape positions)*

  such that

  - $\forall\{s\}.\, s \downarrow \mathsf{o} = s$,
  - $\forall\{s, p, p'\}.\, s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$,
  - $\forall\{s, p\}.\, p \oplus\{s\}\, \mathsf{o} = p$,
  - $\forall\{s, p\}.\, \mathsf{o}\{s\} \oplus p = p$,
  - $\forall\{s, p, p', p''\}.\, (p \oplus\{s\}\, p') \oplus p'' = p \oplus (p' \oplus p'')$.

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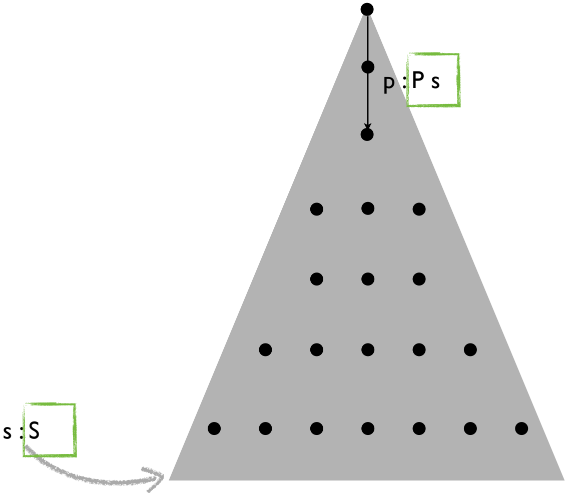
  such that

  - $\forall\{s\}.\, s \downarrow o = s,$
  - $\forall\{s, p, p'\}.\, s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p',$
  - $\forall\{s, p\}.\, p \oplus\{s\}\, o = p,$
  - $\forall\{s, p\}.\, o\{s\} \oplus p = p,$
  - $\forall\{s, p, p', p''\}.\, (p \oplus\{s\}\, p') \oplus p'' = p \oplus (p' \oplus p'').$

Directed containers illustrated



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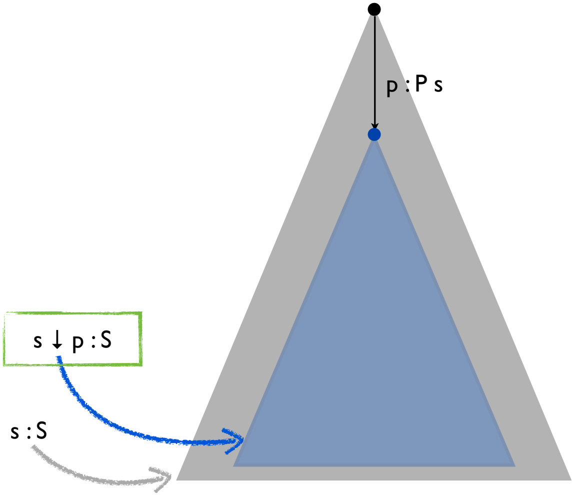

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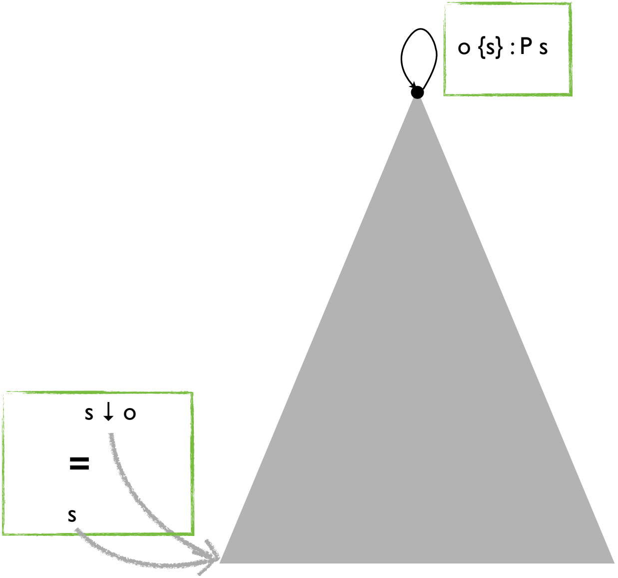

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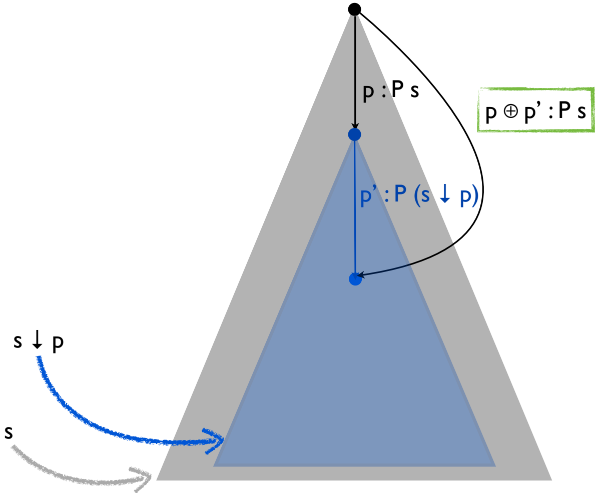

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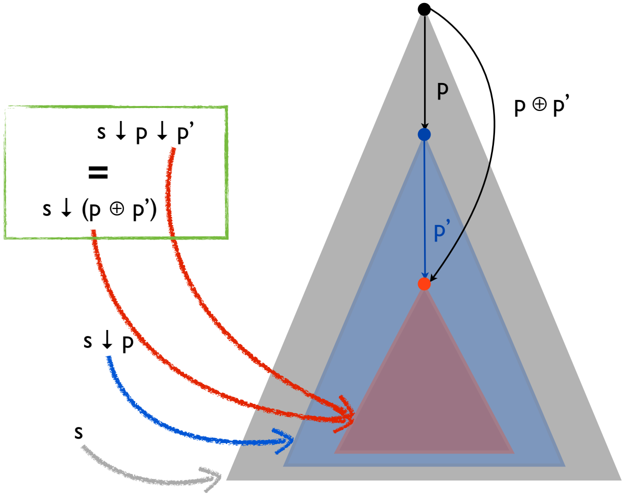

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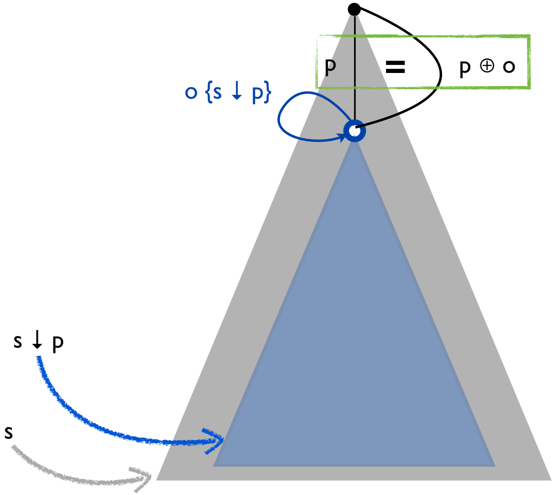

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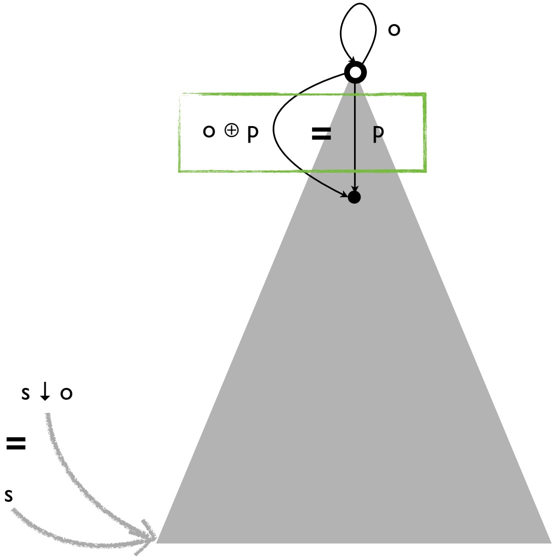

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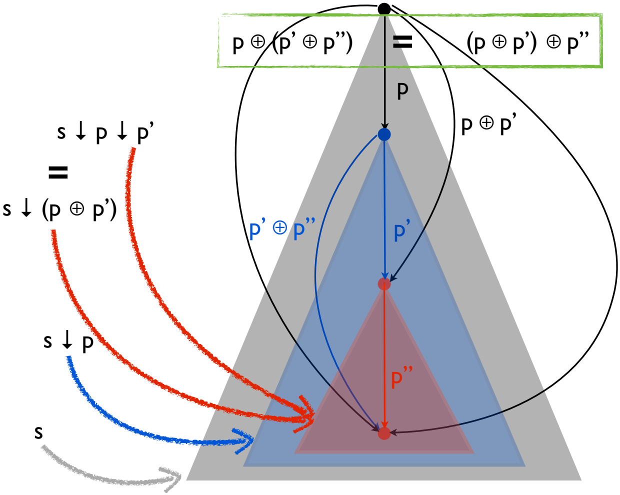
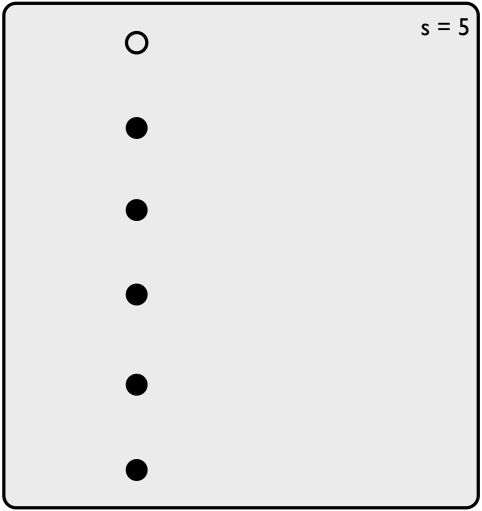


Non-empty lists and streams

- Non-empty lists
  - $S = \mathrm{Nat}$ *(shapes)*
  - $P\, s = [0..s]$ *(positions)*
  - $s \downarrow p = s - p$ *(subshapes)*
  - $\mathrm{o} = 0$ *(root)*
  - $p \oplus \{s\}\, p' = p + p'$ *(subshape positions)*
- Streams are represented similarly with $S = 1$ and $P * = \mathrm{Nat}$

Non-empty lists illustrated



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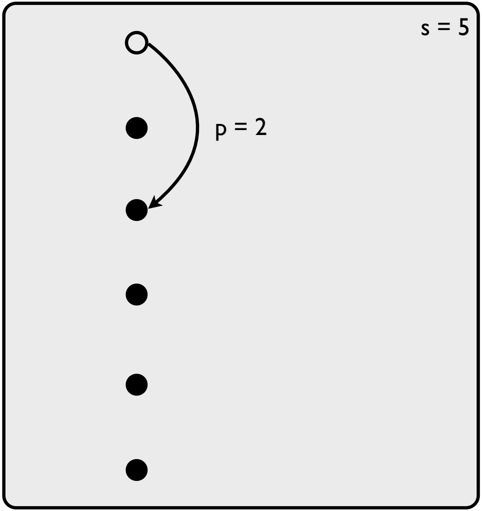

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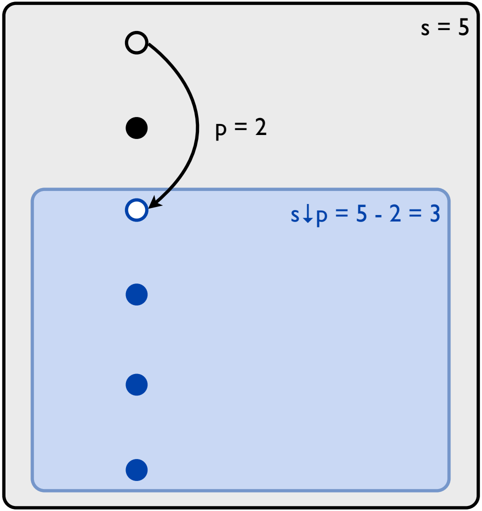

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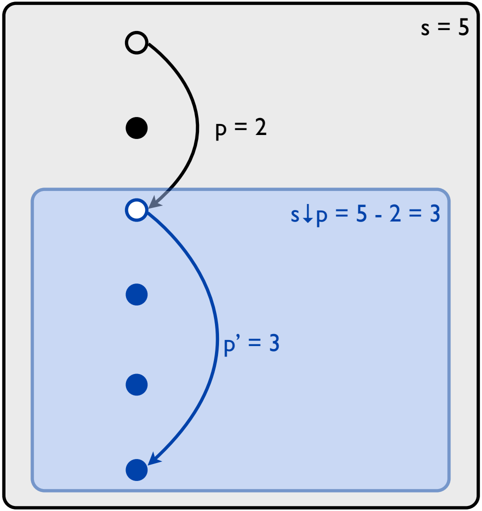

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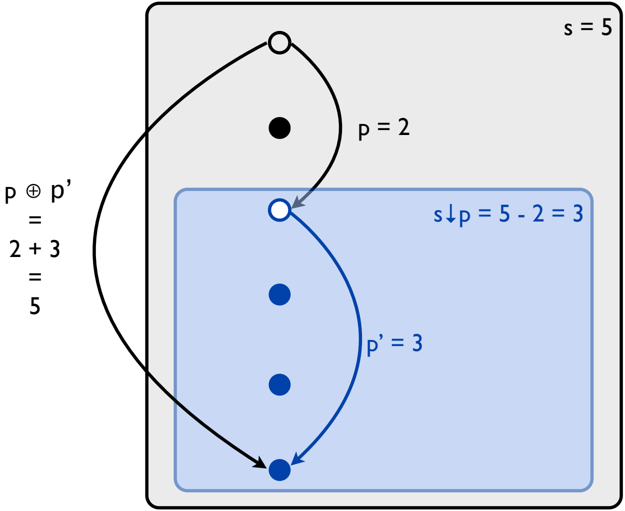


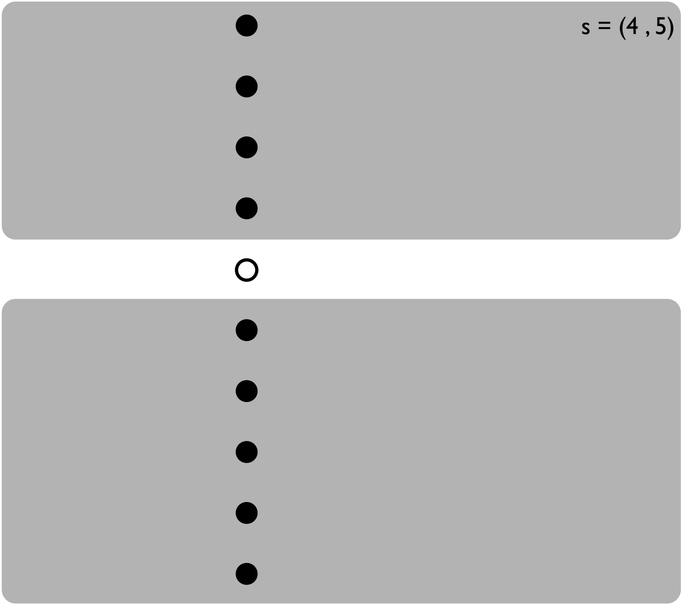
Non-empty lists with a focus

- *Zippers* (tree-like datastructures with a focus position) consist of a context (path from the root to the focus position, with side subtrees) and a focal subtree

- Non-empty lists with a focus
  - $S = \mathrm{Nat} \times \mathrm{Nat}$ *(shapes)*
  - $P\,(s_0, s_1) = [-s_0..s_1] = [-s_0.. -1] \cup [0..s_1]$ *(positions)*

  - $(s_0, s_1) \downarrow p = (s_0 + p, s_1 - p)$ *(subshapes)*
  - $\mathrm{o}\,\{s_0, s_1\} = 0$ *(root)*
  - $p \oplus \{s_0, s_1\}\, p' = p + p'$ *(subshape positions)*

Non-empty list zippers illustrated



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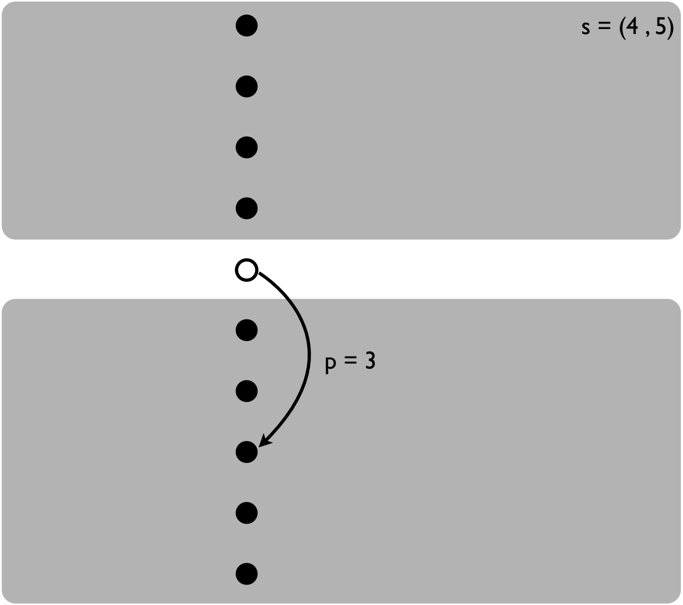

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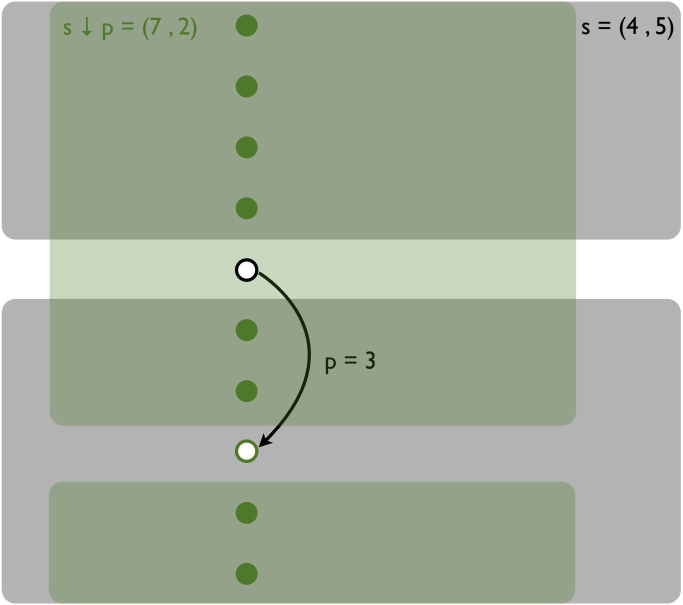

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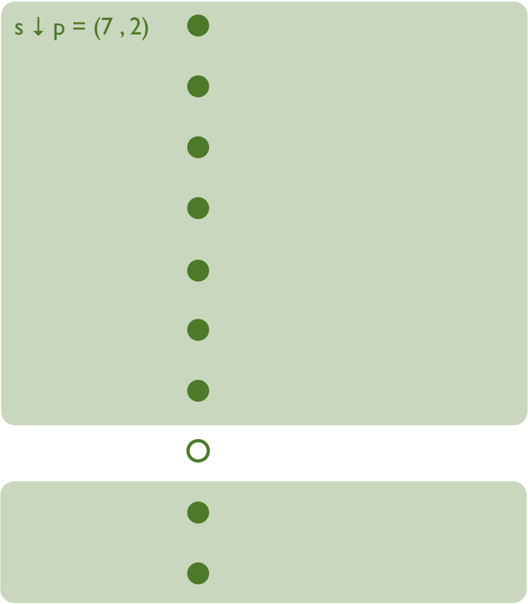

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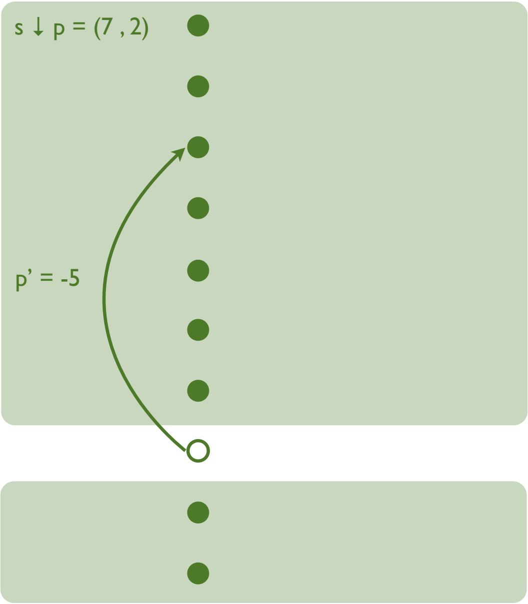

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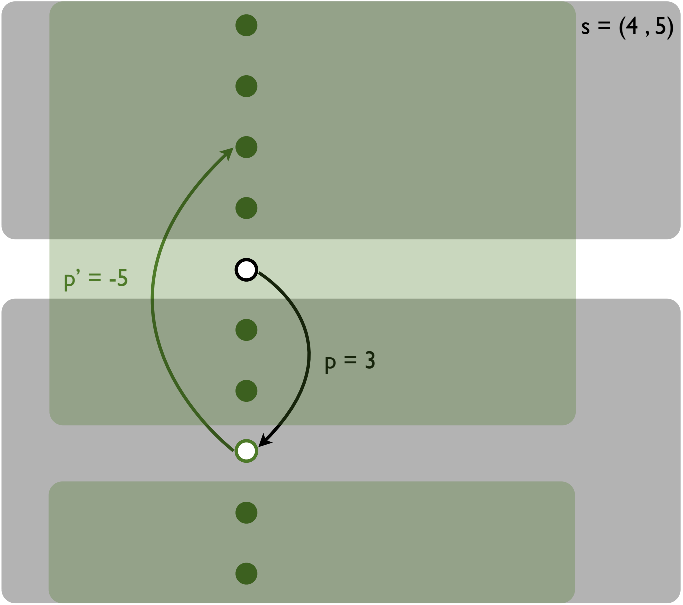

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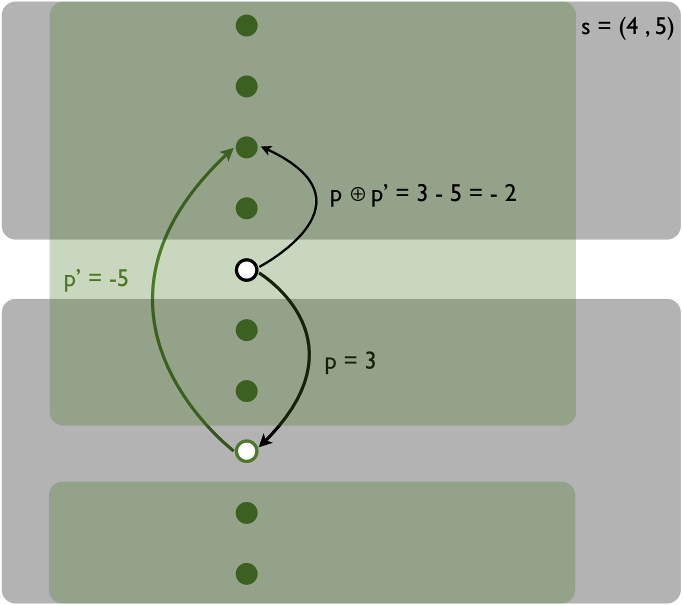

Directed container morphisms

- A directed container morphism $t \lhd q$ between $(S \lhd P, \downarrow, \mathsf{o}, \oplus)$ and $(S' \lhd P', \downarrow', \mathsf{o}', \oplus')$ is given by
  - $t : S \to S'$
  - $q : \Pi\{s : S\}. P'\,(t\,s) \to P\,s$

  such that

  - $\forall\{s, p\}.\, t\,(s \downarrow q\,p) = t\,s \downarrow' p$
  - $\forall\{s\}.\, \mathsf{o}\,\{s\} = q\,(\mathsf{o}'\,\{t\,s\})$
  - $\forall\{s, p, p'\}.\, q\,p \oplus \{s\}\, q\,p' = q\,(p \oplus' \{t\,s\}\, p')$
- Identity $\mathrm{id}^{\mathrm{dc}} = \mathrm{id}\,\{S\} \lhd \lambda\{s\}.\,\mathrm{id}\,\{P\,s\}$
- Composition $(t' \lhd q') \circ^{\mathrm{dc}} (t \lhd q) =$
  $$= (t' \circ t) \lhd (\lambda\{s\}.\, q\,\{s\} \circ q'\,\{t\,s\})$$
- Directed containers form a category **DCont**

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Interpretation (semantics) of directed containers

- Any directed container $(S \lhd P, \downarrow, \mathrm{o}, \oplus)$ defines a functor ~~comonad~~ $[\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}} = (D, \varepsilon, \delta)$ where

  - $D : \mathrm{Set} \to \mathrm{Set}$

    $D\,X = \Sigma s : S.\, P\,s \to X$

    $D\,f\,(s, v) = (s, f \circ v)$

  - $\varepsilon : \forall\{X\}.(\Sigma s : S.\, P\,s \to X) \to X$

    $\varepsilon\,(s, v) = v\,(\mathrm{o}\,\{s\})$

  - $\delta : \forall\{X\}.\,(\Sigma s : S.\, P\,s \to X) \to \Sigma s : S.\, P\,s \to \Sigma s' : S.\, P\,s' \to X$

    $\delta\,(s, v) = (s, \lambda p.\,(s \downarrow p, \lambda p'.\, v\,(p \oplus \{s\}\,p')))$

Interpretation (semantics) of directed containers

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Interpretation of directed container morphisms

- Any directed container morphism $t \lhd q$ between $(S \lhd P, \downarrow, o, \oplus)$ and $(S' \lhd P', \downarrow', o', \oplus')$ defines a natural transformation / ~~comonad morphism~~ $[\![t \lhd q]\!]^{dc}$ between $[\![S \lhd P, \downarrow, o, \oplus]\!]^{dc}$ and $[\![S' \lhd P', \downarrow', o', \oplus']\!]^{dc}$
  - $[\![t \lhd q]\!]^{dc} : \forall \{X\}. (\Sigma s : S. P\, s \to X) \to \Sigma s' : S'. P'\, s' \to X$

    $[\![t \lhd q]\!]^{dc}\, (s, v) = (t\, s, v \circ q\, \{s\})$

- $[\![-]\!]^{dc}$ preserves the identities and composition

- $[\![-]\!]^{dc}$ is a functor from **DCont** to **Endo** / ~~*Cmnds*~~ (**Set**)

Interpretation of directed container morphisms

- Any directed container morphism $t \lhd q$ between $(S \lhd P, \downarrow, \mathrm{o}, \oplus)$ and $(S' \lhd P', \downarrow', \mathrm{o}', \oplus')$ defines a ~~natural transformation~~/comonad morphism $[\![t \lhd q]\!]^{\mathrm{dc}}$ between $[\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}}$ and $[\![S' \lhd P', \downarrow', \mathrm{o}', \oplus']\!]^{\mathrm{dc}}$
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Interpretation is fully faithful

- A natural transformation / ~~comonad morphism~~ τ between $[\![S \lhd P, \downarrow, \mathrm{o}, \oplus]\!]^{\mathrm{dc}}$ and $[\![S' \lhd P', \downarrow', \mathrm{o}', \oplus']\!]^{\mathrm{dc}}$ defines a directed container morphism $\ulcorner \tau \urcorner^{\mathrm{dc}} = (t \lhd q)$ between $(S \lhd P, \downarrow, \mathrm{o}, \oplus)$ and $(S' \lhd P', \downarrow', \mathrm{o}', \oplus')$

    - $t : S \to S'$
      $t\,s = \mathsf{fst}\,(\tau\,\{P\,s\}\,(s, \mathsf{id}))$
    - $q : \Pi\{s : S\}.\, P'\,(t\,s) \to P\,s$
      $q\,\{s\} = \mathsf{snd}\,(\tau\,\{P\,s\}\,(s, \mathsf{id}))$

- $\ulcorner \tau \urcorner^{\mathrm{dc}}$ satisfies,

    - $\ulcorner [\![h]\!]^{\mathrm{dc}} \urcorner^{\mathrm{dc}} = h$
    - $\ulcorner \tau \urcorner^{\mathrm{dc}} = \ulcorner \tau' \urcorner^{\mathrm{dc}}$ implies $\tau = \tau'$

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- $[\![-]\!]^{\mathrm{dc}}$ is a fully faithful functor

Containers $\cap$ comonads $=$ directed containers

- Any comonad (D, ε, δ), such that $D = [\![S \lhd P]\!]^{\mathrm{c}}$, determines a directed container

  $$\lceil (D, \varepsilon, \delta), S \lhd P \rceil = (S \lhd P, \downarrow, \mathsf{o}, \oplus)$$

  where

  - $s \downarrow p = \mathsf{snd}\,(t^{\delta}\, s)\, p$
  - $\mathsf{o}\,\{s\} = q^{\varepsilon}\{s\} *$
  - $p \oplus \{s\}\, p' = q^{\delta}\,\{s\}\,(p, p')$

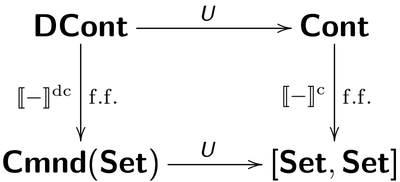
  using the container morphisms

  - $t^{\varepsilon} \lhd q^{\varepsilon} : S \lhd P \to \mathsf{Id}^{\mathrm{c}}$
    $t^{\varepsilon} \lhd q^{\varepsilon} = \ulcorner \mathsf{e} \circ \varepsilon \urcorner^{\mathrm{c}}$
  - $t^{\delta} \lhd q^{\delta} : S \lhd P \to (S \lhd P) \cdot^{\mathrm{c}} (S \lhd P)$
    $t^{\delta} \lhd q^{\delta} = \ulcorner \mathsf{m}\,\{S \lhd P\}\,\{S \lhd P\} \circ \delta \urcorner^{\mathrm{c}}$

- It is forced that
  - $\forall \{s\}.\, t^{\varepsilon}\, s = *$ and $\forall \{s\}.\, \mathsf{fst}\,(t^{\delta}\, s) = s$

Containers ∩ comonads = directed containers ctd.

- For any comonad (D, ε, δ), such that $D = [\![S \lhd P]\!]^{\mathrm{c}}$,
  - $[\![\lceil (D, \varepsilon, \delta), S \lhd P \rceil]\!]^{\mathrm{dc}} = (D, \varepsilon, \delta)$
- For any directed container $(S \lhd P, \downarrow, \mathsf{o}, \oplus)$,
  - $\lceil [\![S \lhd P, \downarrow, \mathsf{o}, \oplus]\!]^{\mathrm{dc}}, S \lhd P \rceil = (S \lhd P, \downarrow, \mathsf{o}, \oplus)$
- The following is a pullback in **CAT**:



Constructions

- Coproduct of directed containers
- Cofree directed containers
- Focussing of a container
- Strict directed containers
- Composition of a strict and non-strict directed container
- Product of strict directed containers
- Distributive laws between directed containers

Conclusion

- Directed containers are a natural notion—cover a natural class of examples and admit an elegant theory
- They give a characterization of containers whose interpretation carries a comonad structure

Questions?

Extra material

Coproduct of directed containers

- Given directed containers $E_0 = (S_0 \lhd P_0, \downarrow_0, \mathrm{o}_0, \oplus_0)$, $E_1 = (S_1 \lhd P_1, \downarrow_1, \mathrm{o}_1, \oplus_1)$, their coproduct is $E = E_0 + E_1$ given as $(S \lhd P, \downarrow, \mathrm{o}, \oplus)$ where
  - $S = S_0 + S_1$
  - $P\,(\mathsf{inl}\,s) = P_0\,s$
    $P\,(\mathsf{inr}\,s) = P_1\,s$
  - $\mathsf{inl}\,s \downarrow p = \mathsf{inl}\,(s \downarrow_0 p)$
    $\mathsf{inr}\,s \downarrow p = \mathsf{inr}\,(s \downarrow_1 p)$
  - $\mathrm{o}\,\{\mathsf{inl}\,s\} = \mathrm{o}_0\,\{s\}$
    $\mathrm{o}\,\{\mathsf{inr}\,s\} = \mathrm{o}_1\,\{s\}$
  - $p \oplus \{\mathsf{inl}\,s\}\,p' = p \oplus_0 \{s\}\,p'$
    $p \oplus \{\mathsf{inr}\,s\}\,p' = p \oplus_1 \{s\}\,p'$
- $[\![E_0 + E_1]\!]^{\mathrm{dc}} \cong [\![E_0]\!]^{\mathrm{dc}} + [\![E_1]\!]^{\mathrm{dc}}$

The cofree directed container

- Given a container $C = (S_0 \lhd P_0)$, the cofree directed container on it is $E = (S \lhd P, \downarrow, \mathsf{o}, \oplus)$ where
  - $S = \nu Z.\, \Sigma s : S_0 . P_0\, s \to Z$,
  - $P = \mu Z.\, \lambda(s, v).\, 1 + \Sigma p : P_0\, s . Z\, (v\, p)$,
  - $\mathsf{o}\, \{s, v\} = \mathsf{inl}\, *$,
  - $(s, v) \downarrow \mathsf{inl}\, * = (s, v)$,
    $(s, v) \downarrow \mathsf{inr}\, (p, p') = v\, p \downarrow p'$,
  - $\mathsf{inl}\, * \oplus \{s, v\}\, p'' = p''$,
    $\mathsf{inr}\, (p, p') \oplus \{s, v\}\, p'' = \mathsf{inr}\, (p, p' \oplus \{v\, p\}\, p'')$.

- $D\, X = \nu Z.\, X \times [\![C]\!]^{\mathrm{c}}\, Z$
  - is the carrier of the cofree comonad on the functor $[\![C]\!]^{\mathrm{c}}$

- Instead of ν, one could also use μ in S, to get the directed container representation of the cofree recursive comonad.

Focussing a container

- Given a container $C_0 = (S_0 \lhd P_0)$, we can focus it by defining a directed container $E = (S \lhd P, \downarrow, \mathsf{o}, \oplus)$ where
  - $S = \Sigma s : S_0.\, P_0\, s$
  - $P\,(s, p) = P_0\, s$
  - $\mathsf{o}\,\{s, p\} = p$
  - $(s, p) \downarrow p' = (s, p')$
  - $p' \oplus \{s, p\}\, p'' = p''$
- Focussed container interprets into the canonical comonad structure on $\partial [\![C]\!]^{\mathrm{c}} \times \mathsf{Id}$ where ∂F denotes the derivative of F

Directed container from a monoid

- Any monoid $(M, e, \bullet)$ gives a directed container $E = (S \lhd P, \downarrow, \mathrm{o}, \oplus)$ by
  - $S = 1$
  - $P * = M$
  - $* \downarrow p = *$
  - $\mathrm{o}\{*\} = e$
  - $p \oplus \{*\} p' = p \bullet p'$

- $[\![E]\!]^{\mathrm{dc}}\, X = \Sigma s : *.\, M \to X \cong M \to X$

Containers $\cap$ Monads $=$?

- Given a container $C = (S \lhd P)$
- The structure (η, μ) of a monad on $[\![S \lhd P]\!]^{\mathrm{c}}$ could be represented as
  - $\mathsf{e} : S$ *(for the shape map for η)*
  - $\bullet : \Pi s : S.(P\,s \to S) \to S$ *(for the shape map for μ)*
  - $\nwarrow : \Pi\{s : S\}.\,\Pi v : P\,s \to S.P\,(s \bullet v) \to P\,s$
    (for the position map for μ)
  - $\nearrow : \Pi\{s : S\}.\,\Pi v : P\,s \to S.$
    $\Pi p : P\,(s \bullet v).\,P\,(v\,(v \nwarrow \{s\} p))$
    (for the position map for μ)