Distributive Laws of Directed Containers
[Extended Abstract]

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Containers [1] are an elegant representation of a wide class of datatypes in terms of shapes and positions in shapes. In our FoSSaCS 2012 work [2], we introduced directed containers as a special case to account for the common situation where every position in a shape determines another shape, informally the subshape rooted by that position; some examples being the datatypes of nonempty lists and trees and the corresponding zipper datatypes. While containers interpret into set functors via a fully faithful monoidal functor, directed containers interpret into comonads. Further, it is also true that every comonad whose underlying functor is a container is represented by a directed container. In this paper, we develop a characterization of distributive laws between such comonads.

A container $S \lhd P$ is given by a set S (of shapes) and a shape-indexed family $P : S \to \mathsf{Set}$ (of positions). A morphism between containers $S \lhd P$ and $S' \lhd P'$ is a pair $t \lhd q$ of maps $t : S \to S'$ and $q : \Pi\{s : S\}.\, P'\,(t\,s) \to P\,s$. (We use Agda's syntax of braces for implicit arguments.) Containers form a category **Cont** carrying a monoidal structure defined by $\mathsf{Id}^{\mathrm{c}} = 1 \lhd \lambda * .\, 1$ and $(S_0 \lhd P_0) \cdot^{\mathrm{c}} (S_1 \lhd P_1) = \Sigma s : S_0.\, P_0\, s \to S_1 \lhd \lambda\,(s, v).\, \Sigma p_0 : P_0\, s.\, P_1\,(v\, p_0)$ together suitable unital and associativity laws.

The interpretation of a container $S \lhd P$ is the set functor given by $[\![S \lhd P]\!]^{\mathrm{c}}\, X = \Sigma s : S.\, P\, s \to X$, $[\![S \lhd P]\!]^{\mathrm{c}}\, f\,(s, v) = (s, f \circ v)$. The interpretation of a container map $t \lhd q$ is the natural transformation $[\![t \lhd q]\!]^{\mathrm{c}}(s, v) = (t\, s, v \circ q\,\{s\})$. $[\![-]\!]^{\mathrm{c}}$ is a fully faithful monoidal functor from **Cont** to $[\mathbf{Set}, \mathbf{Set}]$.

A directed container is a container $S \lhd P$ together with three operations

- $\downarrow : \Pi s : S.\, P\, s \to S$ (the subshape for a position),
- $\mathsf{o} : \Pi\{s : S\}.\, P\, s$ (the root),
- $\oplus : \Pi\{s : S\}.\, \Pi p : P\, s.\, P\,(s \downarrow p) \to P\, s$ (translation of subshape positions into positions in the global shape),

satisfying the following two shape equations and three position equations:

1. $\forall\{s\}.\, s \downarrow \mathsf{o} = s$,
2. $\forall\{s, p, p'\}.\, s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$,

3. $\forall\{s,p\}.\, p \oplus \{s\}\, \mathsf{o} = p$,
4. $\forall\{s,p\}.\, \mathsf{o}\{s\} \oplus p = p$,
5. $\forall\{s,p,p',p''\}.\, (p \oplus \{s\}\, p') \oplus p'' = p \oplus (p' \oplus p'')$.

Notice that, modulo the fact that the positions involved come from different sets, equations 3-5 are the equations of a monoid. Equations 1-2 make sure that equations 4-5 are well-typed. A morphism between directed containers $(S \lhd P, \downarrow, \mathsf{o}, \oplus)$ and $(S' \lhd P', \downarrow', \mathsf{o}', \oplus')$ is a morphism $t \lhd q$ between the containers $S \lhd P$ and $S' \lhd P'$ that satisfies these equations:

1. $\forall\{s,p\}.\, t\,(s \downarrow q\, p) = t\, s \downarrow' p$,
2. $\forall\{s\}.\, \mathsf{o}\,\{s\} = q\,(\mathsf{o}'\,\{t\, s\})$,
3. $\forall\{s,p,p'\}.\, q\, p \oplus \{s\}\, q\, p' = q\,(p \oplus' \{t\, s\}\, p')$.

Here, equations 2-3 are reminiscent of the equations of a monoid morphism. Directed containers form a category **DCont**.

The interpretation $[\![S \lhd P, \downarrow, \mathsf{o}, \oplus]\!]^{\mathrm{dc}}$ of a directed container is the set functor $[\![S \lhd P]\!]^{\mathrm{c}}$ together with natural transformations ε, δ where $\varepsilon\,(s, v) = v\,(\mathsf{o}\,\{s\})$ and $\delta\,(s,v) = (s, \lambda p.\, (s \downarrow p, \lambda p'.\, v\,(p \oplus \{s\}\, p')))$, making a comonad. The interpretation $[\![t \lhd q]\!]^{\mathrm{dc}}$ of a directed container morphism is $[\![t \lhd q]\!]^{\mathrm{c}}$, which is a comonad morphism. $[\![-]\!]^{\mathrm{dc}}$ is a fully-faithful functor $\mathbf{DCont} \to \mathbf{Comonads}(\mathbf{Set})$. Moreover, every comonad whose underlying functor is a container is represented by a directed container. Actually, **DCont** is isomorphic to $\mathbf{Comonoids}(\mathbf{Cont})$), and that in turn is easily seen to be the pullback of $U : \mathbf{Comonads}(\mathbf{Set}) \to [\mathbf{Set}, \mathbf{Set}]$ along $[\![-]\!]^{\mathrm{c}} : \mathbf{Cont} \to [\mathbf{Set}, \mathbf{Set}]$.

A sufficient condition for the composition of the underlying functors of two comonads to carry a comonad structure is that they distribute over each other. We develop the corresponding concept for directed containers and show that it is adequate.

For two directed containers $(S_0 \lhd P_0, \downarrow_0, \mathsf{o}_0, \oplus_0)$ and $(S_1 \lhd P_1, \downarrow_1, \mathsf{o}_1, \oplus_1)$, we define a distributive law to be given by operations

– $t_1 : \Pi s : S_0.\, \Pi v : P_0\, s \to S_1.\, P_1\,(v\,(\mathsf{o}_0\,\{s\})) \to S_0$,
– $q_0 : \Pi\{s : S_0\}.\, \Pi\{v : P_0\, s \to S_1\}.\, \Pi p_1 : P_1\,(v\,(\mathsf{o}_0\,\{s\})).$
$\qquad P_0\,(t_1\, s\, v\, p_1)) \to P_0\, s$,
– $q_1 : \Pi\{s : S_0\}.\, \Pi\{v : P_0\, s \to S_1\}.\, \Pi p_1 : P_1\,(v\,(\mathsf{o}_0\,\{s\})).$
$\qquad \Pi p_0 : P_0\,(t_1\, s\, v\, p_1).\, P_1\,(v\,(q_0\,\{s\}\,\{v\}\, p_1\, p_0))$

satisfying the equations

1. $\forall\{s,v,p_1,p_0\}.\, t_1\, s\, v\, p_1 \downarrow_0 p_0$
$\qquad = t_1\,(s \downarrow_0 q_0\, p_1\, p_0)\,(\lambda p_0'.\, v\,(q_0\, p_1\, p_0 \oplus_0 p_0'))\,(q_1\, p_1\, p_0)$,
2. $\forall\{s,v\}.\, t_1\, s\, v\, \mathsf{o}_1 = s$,
3. $\forall\{s,v,p_1,p_1'\}.\, t_1\, s\, v\,(p_1 \oplus_1 p_1') = t_1\,(t_1\, s\, v\, p_1)\,(\lambda p_0.\, v\,(q_0\, p_1\, p_0) \downarrow_1 q_1\, p_1\, p_0)\, p_1'$,

4. $\forall\{s,v,p_1\}.\, q_0\,\{s\}\,\{v\}\, p_1\, \mathsf{o}_0 = \mathsf{o}_0$,
5. $\forall\{s,v,p_1,p_0,p_0'\}.\, q_0\,\{s\}\,\{v\}\, p_1\,(p_0 \oplus_0 p_0') = q_0\, p_1\, p_0 \oplus_0 q_0\,(q_1\, p_1\, p_0)\, p_0'$,
6. $\forall\{s,v,p_0\}.\, q_0\,\{s\}\,\{v\}\, \mathsf{o}_1\, p_0 = p_0$,
7. $\forall\{s,v,p_1,p_1',p_0\}.\, q_0\,\{s\}\,\{v\}\,(p_1 \oplus_1 p_1')\, p_0 = q_0\, p_1\,(q_0\, p_1'\, p_0)$,

8. $\forall \{s, v, p_1\}.\, q_1\,\{s\}\,\{v\}\,p_1\,\mathsf{o}_0 = p_1$,
9. $\forall \{s, v, p_1, p_0, p'_0\}.\, q_1\,\{s\}\,\{v\}\,p_1\,(p_0 \oplus_0 p'_0) = q_1\,(q_1\,p_1\,p_0)\,p'_0$,
10. $\forall \{s, v, p_0\}.\, q_1\,\{s\}\,\{v\}\,\mathsf{o}_1\,p_0 = \mathsf{o}_1$,
11. $\forall \{s, v, p_1, p'_1, p_0\}.\, q_1\,\{s\}\,\{v\}\,(p_1 \oplus_1 p'_1)\,p_0 = q_1\,p_1\,(q_0\,p'_1\,p_0) \oplus_1 q_1\,p'_1\,p_0$.

If we ignore that both P_0 and P_1 are families rather than sets (i.e., confine ourselves to the special case $S_0 = S_1 = 1$), the equations 4-11 are the equations required of two monoids to have a knit or Zappa-Szép product (see [3, Lemma 3.13 (xv)]).

A distributive law as above determines a container morphism $t \lhd q : (S_0 \lhd P_0) \cdot^{\mathrm{c}} (S_1 \lhd P_1) \to (S_1 \lhd P_1) \cdot^{\mathrm{c}} (S_0 \lhd P_0)$ by $t\,(s, v) = (v\,(\mathsf{o}_0\,\{s\}), t_1\,s\,v)$ and $q\,\{s, v\}\,(p_1, p_0) = (q_0\,\{s\}\,\{v\}\,p_1\,p_0, q_1\,\{s\}\,\{v\}\,p_1\,p_0)$. The interpreting natural transformation $[\![t \lhd q]\!]^{\mathrm{c}}$ gives a distributive law θ between the comonads $[\![S_0 \lhd P_0, \downarrow_0, \mathsf{o}_0, \oplus_0]\!]^{\mathrm{dc}}$ and $[\![S_1 \lhd P_1, \downarrow_1, \mathsf{o}_1, \oplus_1]\!]^{\mathrm{dc}}$ by $\theta\,(s, v) = (\pi_0\,(v\,(\mathsf{o}_0\,\{s\})), \lambda p_1.\,(t_1\,s\,(\pi_0 \circ v)\,p_1, \lambda p_0.\,\pi_1\,(v\,(q_0\,(p_1, p_0)))\,(q_1\,(p_1, p_0))))$. And conversely, any distributive law between these two comonads corresponds to a distributive law between the two directed containers. The fact that the composition of two directed containers distributing over each other is a directed container follows from the properties of $[\![-]\!]^{\mathrm{dc}}$ ("via the semantics"), but is also easily proved directly ("syntactically").

We see that, just as comonads whose underlying functor is the interpretation of a container have some special properties (the outer shape of the nested datastructure returned by the comultiplication is the shape of the given datastructure), so do distributive laws between such comonads have some similar properties (the outer shape of the nested datastructure returned by the distributive law is the inner shape at the outer root position of the given nested datastructure).

In the paper, we present and analyze several generic distributive laws of comonads (e.g., distributivity of any comonad over the product comonad, distributive laws for cofree comonads) in this form as well as some that are specific to comonads whose underlying functors are containers.

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